

# A Characterization of Inner Product Spaces Related to the Skew-Angular Distance

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## Abstract

A new refinement of the triangle inequality is presented in normed linear spaces. Moreover, a simple characterization of inner product spaces is obtained by using the skew-angular distance.

**Keywords:** Triangle inequality, characterization of inner product spaces, angular distance

## 1 Introduction

In 2006, Maligranda [1, Theorem 1] (see also [2]) introduced the following strengthening of the triangle inequality and its reverse: For any nonzero vectors  $x$  and  $y$  in a real normed linear space  $X = (X, \|\cdot\|)$  it is true that

$$\|x + y\| \leq \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \min\{\|x\|, \|y\|\} \quad (1.1)$$

and

$$\|x + y\| \geq \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \max\{\|x\|, \|y\|\}. \quad (1.2)$$

Also, the author used (1.1) and (1.2) for the following estimation of the *angular distance*  $\alpha[x, y] = \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$  between two nonzero elements  $x$  and  $y$  in  $X$  which was defined by Clarkson in [3].

$$\frac{\|x - y\| - |\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}} \leq \alpha[x, y] \leq \frac{\|x - y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}}. \quad (1.3)$$

The right hand of estimate (1.3) is a refinement of the Massera-Schaffer inequality proved in 1958 (see [4, Lemma 5.1]): for nonzero vectors  $x$  and  $y$  in  $X$  we have that  $\alpha[x, y] \leq$

$\frac{2\|x-y\|}{\max\{\|x\|, \|y\|\}}$ , which is stronger than the Dunkl-Williams inequality  $\alpha[x, y] \leq \frac{4\|x-y\|}{\|x\|+\|y\|}$  proved in [5]. In the same paper, Dunkl and Williams proved that the constant 4 can be replaced by 2 if and only if  $X$  is an inner product space.

The main aim of this paper is to obtain a new and simple characterization of inner product spaces. To proceed in this direction we first present a refinement of the triangle inequality in normed linear spaces and introduce the notion of skew-angular distance. Next, we compare the angular distance and skew-angular distance with each other.

## 2 A refinement of triangle inequality

we start with the following strengthening of the triangle inequality.

**Theorem 2.1** *For nonzero vectors  $x$  and  $y$  in a real normed linear space  $X = (X, \|\cdot\|)$  we have*

$$\|x + y\| \leq \|x\| + \|y\| - \left( \frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \left\| \frac{x}{\|y\|} + \frac{y}{\|x\|} \right\| \right) \min\{\|x\|, \|y\|\} \quad (2.1)$$

and

$$\|x + y\| \geq \|x\| + \|y\| - \left( \frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \left\| \frac{x}{\|y\|} + \frac{y}{\|x\|} \right\| \right) \max\{\|x\|, \|y\|\}. \quad (2.2)$$

**Proof.** Without loss of generality we may assume that  $\|x\| \leq \|y\|$ . Then, by the triangle inequality,

$$\begin{aligned} \|x + y\| &= \left\| \frac{\|x\|}{\|y\|}x + \frac{\|x\|}{\|x\|}y + \left(1 - \frac{\|x\|}{\|y\|}\right)x \right\| \\ &\leq \|x\| \left\| \frac{x}{\|y\|} + \frac{y}{\|x\|} \right\| + \|x\| - \frac{\|x\|^2}{\|y\|} \\ &= \|x\| + \|x\| \left( \left\| \frac{x}{\|y\|} + \frac{y}{\|x\|} \right\| - \frac{\|x\|}{\|y\|} \right) \\ &= \|x\| + \|y\| + \|x\| \left( \left\| \frac{x}{\|y\|} + \frac{y}{\|x\|} \right\| - \frac{\|x\|}{\|y\|} - \frac{\|y\|}{\|x\|} \right) \end{aligned}$$

which establishes estimate (2.1). Similarly, the computation

$$\begin{aligned} \|x + y\| &= \left\| \frac{\|y\|}{\|y\|}x + \frac{\|y\|}{\|x\|}y + \left(1 - \frac{\|y\|}{\|x\|}\right)y \right\| \\ &\geq \|y\| \left\| \frac{x}{\|y\|} + \frac{y}{\|x\|} \right\| - \left( \frac{\|y\|^2}{\|x\|} - \|y\| \right) \\ &= \|y\| + \|y\| \left( \left\| \frac{x}{\|y\|} + \frac{y}{\|x\|} \right\| - \frac{\|y\|}{\|x\|} \right) \\ &= \|x\| + \|y\| + \|y\| \left( \left\| \frac{x}{\|y\|} + \frac{y}{\|x\|} \right\| - \frac{\|x\|}{\|y\|} - \frac{\|y\|}{\|x\|} \right) \end{aligned}$$

gives inequality (2.2).  $\square$

The following examples show that neither our refinement nor Maligranda's refinement of the triangle inequality is always better.

**Example 2.2** Let  $X$  be the normed space  $\mathbb{R}$  with the norm  $\|x\| = |x|$ . Then, for  $x = 1$  and  $y = -2$  we have

$$2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| = 2 > 1 = \frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \left\| \frac{x}{\|y\|} + \frac{y}{\|x\|} \right\|.$$

**Example 2.3** Let  $X = \mathbb{R}^2$  with the norm of  $x = (a, b)$  given by  $\|x\| = |a| + |b|$ . Take  $x = (3/4, 3/4)$  and  $y = (-1, 0)$ , then  $\|x\| = 3/2$ ,  $\|y\| = 1$ . Therefore,

$$\frac{x}{\|x\|} = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \frac{y}{\|y\|} = (-1, 0), \quad \frac{x}{\|y\|} = \left(\frac{3}{4}, \frac{3}{4}\right), \quad \frac{y}{\|x\|} = \left(-\frac{2}{3}, 0\right)$$

and

$$2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| = 1 < \frac{4}{3} = \frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \left\| \frac{x}{\|y\|} + \frac{y}{\|x\|} \right\|.$$

We can gather estimates (2.1) and (2.2) together as

$$\begin{aligned} \|x + y\| + \left( \frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \left\| \frac{x}{\|y\|} + \frac{y}{\|x\|} \right\| \right) \min\{\|x\|, \|y\|\} &\leq \|x\| + \|y\| \\ &\leq \|x + y\| + \left( \frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \left\| \frac{x}{\|y\|} + \frac{y}{\|x\|} \right\| \right) \max\{\|x\|, \|y\|\}. \end{aligned}$$

Also, we use them as the estimates for a distance in normed linear spaces which we call *skew-angular distance*.

**Definition 2.4** For two nonzero elements  $x$  and  $y$  in a real normed linear space  $X = (X, \|\cdot\|)$  the distance

$$\beta[x, y] = \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \quad (2.3)$$

is called *skew-angular distance* between  $x$  and  $y$ .

**Corollary 2.5** For any nonzero elements  $x$  and  $y$  in a real normed linear space  $X = (X, \|\cdot\|)$  we have

$$\beta[x, y] \leq \frac{\|x - y\|}{\max\{\|x\|, \|y\|\}} + \frac{|\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}} \quad (2.4)$$

and

$$\beta[x, y] \geq \frac{\|x - y\|}{\min\{\|x\|, \|y\|\}} - \frac{|\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}} \quad (2.5)$$

**Proof.** Estimate (2.2) implies that

$$\left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \max\{\|x\|, \|y\|\} \leq \|x - y\| - \|x\| - \|y\| + \left( \frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} \right) \max\{\|x\|, \|y\|\}.$$

Without loss of generality we may assume that  $\|x\| \leq \|y\|$ . Then

$$\left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \|y\| \leq \|x - y\| + \frac{\|y\|}{\|x\|} (\|y\| - \|x\|)$$

and so

$$\left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \leq \frac{\|x - y\|}{\|y\|} + \frac{|\|y\| - \|x\||}{\|x\|}.$$

Similarly, inequality (2.1) gives that

$$\left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \|x\| \geq \|x - y\| - \frac{\|x\|}{\|y\|} (\|y\| - \|x\|)$$

and so

$$\left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \geq \frac{\|x - y\|}{\|x\|} - \frac{|\|y\| - \|x\||}{\|y\|}.$$

This completes the proof.  $\square$

Estimates (2.1) and (2.2) mean for the skew-angular distance that

$$\frac{\|x - y\|}{\min\{\|x\|, \|y\|\}} - \frac{|\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}} \leq \beta[x, y] \leq \frac{\|x - y\|}{\max\{\|x\|, \|y\|\}} + \frac{|\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}}.$$

Since  $|\|x\| - \|y\|| \leq \|x - y\|$ , we obtain the estimate

$$\beta[x, y] \leq \left( \frac{1}{\max\{\|x\|, \|y\|\}} + \frac{1}{\min\{\|x\|, \|y\|\}} \right) \|x - y\| = \left( \frac{1}{\|x\|} + \frac{1}{\|y\|} \right) \|x - y\|. \quad (2.6)$$

The constant 1 in the estimate (2.6) is the best possible even for an inner product space. In fact, consider  $X = \mathbb{R}$  with the norm of  $x$  given by  $\|x\| = |x|$ . Take  $x = -1$  and  $y = \epsilon$ , where  $\epsilon > 0$  is small. Then

$$\beta[x, y] = \epsilon + \frac{1}{\epsilon} \quad \text{and} \quad \left( \frac{1}{\|x\|} + \frac{1}{\|y\|} \right) \|x - y\| = \left( 1 + \frac{1}{\epsilon} \right) (1 + \epsilon).$$

Hence

$$\beta[x, y] \frac{\|x\| \|y\|}{(\|x\| + \|y\|) \|x - y\|} = \frac{1 + \epsilon^2}{(1 + \epsilon)^2} \rightarrow 1$$

as  $\epsilon \rightarrow 0^+$ .

### 3 Characterization of inner product spaces

In this section we compare the norm-angular distance  $\alpha[x, y]$  with the skew-angular distance  $\beta[x, y]$ . The next theorem due to Lorch will be useful in the sequel.

**Theorem 3.1** (See [6].) *Let  $(X, \|\cdot\|)$  be a real normed linear space. Then the following statements are mutually equivalent:*

- (i) *For each  $x, y \in X$  if  $\|x\| = \|y\|$ , then  $\|x + y\| \leq \|\gamma x + \gamma^{-1}y\|$  (for all  $\gamma \neq 0$ ).*
- (ii) *For each  $x, y \in X$  if  $\|x + y\| \leq \|\gamma x + \gamma^{-1}y\|$  (for all  $\gamma \neq 0$ ), then  $\|x\| = \|y\|$ .*
- (iii)  *$(X, \|\cdot\|)$  is an inner product space.*

**Theorem 3.2** *Let  $(X, \|\cdot\|)$  be a real normed linear space. Then  $(X, \|\cdot\|)$  is an inner product space if and only if for each nonzero elements  $x$  and  $y$  in  $X$ ,*

$$\alpha[x, y] \leq \beta[x, y]. \quad (3.1)$$

**Proof.** Let  $X = (X, \langle \cdot, \cdot \rangle)$  be inner product space,  $x, y \in X$  and  $x, y \neq 0$ . We consider that

$$\begin{aligned} \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\|^2 - \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 &= \left\langle \frac{x}{\|y\|} - \frac{y}{\|x\|}, \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\rangle - \left\langle \frac{x}{\|x\|} - \frac{y}{\|y\|}, \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\rangle \\ &= \frac{\|x\|^2}{\|y\|^2} + \frac{\|y\|^2}{\|x\|^2} - \frac{2\langle x, y \rangle}{\|x\|\|y\|} - \left( \frac{\|x\|^2}{\|x\|^2} + \frac{\|y\|^2}{\|y\|^2} - \frac{2\langle x, y \rangle}{\|x\|\|y\|} \right) \\ &= \frac{\|x\|^2}{\|y\|^2} + \frac{\|y\|^2}{\|x\|^2} - 2 = \left( \frac{\|x\|}{\|y\|} - \frac{\|y\|}{\|x\|} \right)^2 \geq 0. \end{aligned}$$

This proves the necessity.

To prove the sufficiency let  $x, y \in X$ ,  $\|x\| = \|y\|$  and  $\gamma \neq 0$ . From Theorem 3.1 it is enough to prove that  $\|x + y\| \leq \|\gamma x + \gamma^{-1}y\|$ . If  $x = 0$  or  $y = 0$ , then the proof is clear. Let  $x \neq 0$ ,  $y \neq 0$  and  $\gamma > 0$ . Applying inequality (3.1) to  $\gamma^{\frac{1}{2}}x$  and  $-\gamma^{-\frac{1}{2}}y$  instead of  $x$  and  $y$ , respectively, we obtain

$$\left\| \frac{\gamma^{\frac{1}{2}}x}{\gamma^{\frac{1}{2}}\|x\|} + \frac{\gamma^{-\frac{1}{2}}y}{\gamma^{-\frac{1}{2}}\|y\|} \right\| \leq \left\| \frac{\gamma^{\frac{1}{2}}x}{\gamma^{-\frac{1}{2}}\|y\|} + \frac{\gamma^{-\frac{1}{2}}y}{\gamma^{\frac{1}{2}}\|x\|} \right\|.$$

Thus

$$\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \leq \left\| \frac{\gamma x}{\|y\|} + \frac{\gamma^{-1}y}{\|x\|} \right\|.$$

Since  $\|x\| = \|y\| \neq 0$ , then

$$\|x + y\| \leq \|\gamma x + \gamma^{-1}y\|.$$

Now, let  $\gamma$  be negative. Put  $\mu = -\gamma > 0$ . From the positive case we get

$$\|x + y\| \leq \|\mu x + \mu^{-1}y\| = \|\gamma x + \gamma^{-1}y\|.$$

This completes the proof.  $\square$

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